

# Graphs and Hermitian matrices: exact interlacing

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## Abstract

We prove conditions for equality between the extreme eigenvalues of a matrix and its quotient. In particular, we give a lower bound on the largest singular value of a matrix and generalize a result of Finck and Grohmann about the largest eigenvalue of a graph.

**Keywords:** extreme eigenvalues, tight interlacing, graph Laplacian, singular values, nonnegative matrix

## 1 Introduction

Our notation is standard (e.g., see [1], [3], and [6]); in particular, all graphs are defined on the vertex set  $[n] = \{1, \dots, n\}$  and  $G(n)$  stands for a graph of order  $n$ . Given a graph  $G = G(n)$ ,  $\mu_1(G) \geq \dots \geq \mu_n(G)$  are the eigenvalues of its adjacency matrix  $A(G)$ , and  $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G)$  are the eigenvalues of its Laplacian  $L(G)$ . If  $X, Y \subset V(G)$  are disjoint sets, we write  $G[X]$  for the graph induced by  $X$ , and  $G[X, Y]$  for the bipartite graph induced by  $X$  and  $Y$ ; we set  $e(X) = e(G[X])$  and  $e(X, Y) = e(G[X, Y])$ . We assume that partitions consist of nonempty sets.

In this note we study conditions for finding exact eigenvalues using interlacing.

As proved in [2], if  $G = G(n)$  and  $[n] = \bigcup_{i=1}^k P_i$  is a partition, then

$$\mu_1(G) + \dots + \mu_k(G) \geq \sum_{i=1}^k \frac{2e(P_i)}{|P_i|}, \quad (1)$$

$$\mu_{n-k+2}(G) + \dots + \mu_n(G) \leq \sum_{i=1}^k \frac{2e(P_i)}{|P_i|} - \frac{2e(G)}{n}, \quad (2)$$

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$$\lambda_2(G) + \dots + \lambda_k(G) \leq \sum_{1 \leq i < j \leq k} e(P_i, P_j) \left( \frac{1}{|P_i|} + \frac{1}{|P_j|} \right), \quad (3)$$

$$\lambda_{n-k+1}(G) + \dots + \lambda_n(G) \geq \sum_{1 \leq i < j \leq k} e(P_i, P_j) \left( \frac{1}{|P_i|} + \frac{1}{|P_j|} \right). \quad (4)$$

To warm up we shall give necessary conditions for equality in these inequalities.

Call a bipartite graph *semiregular* if the vertices of the same vertex class have the same degree. Call a partition  $V(G) = \bigcup_{i=1}^k P_i$  *semiequitable for  $G$*  if  $G[P_i, P_j]$  is semiregular for  $1 \leq i < j \leq k$ , and *equitable for  $G$*  if, in addition,  $G[P_i]$  is regular for  $i \in [k]$ .

**Theorem 1** *If equality holds in (1) or (2), then the partition  $[n] = \bigcup_{i=1}^k P_i$  is equitable for  $G$ ; moreover, if equality holds in (2), then  $G$  is regular. If equality holds in (3) or (4), then the partition  $[n] = \bigcup_{i=1}^k P_i$  is semiequitable for  $G$ .*

In order to discuss this result from a more general viewpoint, we introduce additional notation and definitions. We order the eigenvalues of an  $n \times n$  Hermitian matrix  $A$  as  $\mu_1(A) \geq \dots \geq \mu_n(A)$ .

Suppose  $1 < k < n$  and let  $A$  and  $B$  be Hermitian matrices of size  $n \times n$  and  $k \times k$ . As usual, we say that the eigenvalues of  $A$  and  $B$  are *interlaced*, if  $\mu_i(A) \geq \mu_i(B) \geq \mu_{n-k+i}(A)$  for all  $i \in [k]$ . The interlacing is called *tight* if there exists an integer  $r \in [0, k]$  such that

$$\mu_i(A) = \mu_i(B) \text{ for } 0 \leq i \leq r \text{ and } \mu_{n-k+i}(A) = \mu_i(B) \text{ for } r < i \leq k.$$

When we must indicate the value  $r$ , we say that the interlacing is  $r$ -tight.

Note that inequalities (1) - (4) are proved using eigenvalue interlacing; we shall see that equality in either of them implies tight interlacing, in turn, implying the conditions of Theorem 1. Hence, the following question arises.

**Question 2** *For which graphs conditions similar to those in Theorem 1 imply tight interlacing.*

Below we answer a simple, yet important case of this question.

**Theorem 3** *Let  $G = G(n)$  and  $[n] = \bigcup_{i=1}^k P_i$  be a partition such that, for all  $i \in [k]$ ,  $G[P_i]$  is empty and, for all  $1 \leq i < j \leq k$ ,  $G[P_i, P_j]$  is empty or complete. Then equality holds in (1), (3), and (4). If  $G$  is regular, equality holds in (2) as well.*

The general case of Question 2 seems rather difficult; however, most often we are interested in simpler problems, which, for convenience, we state for matrices.

We first relax the concept of tight interlacing. Suppose  $1 < k < n$  and let  $A$  and  $B$  be Hermitian matrices of size  $n \times n$  and  $k \times k$  with interlaced eigenvalues. Call the interlacing *exact* if there exist integers  $p, q$  such that  $0 < p + q \leq k$  and

$$\mu_i(A) = \mu_i(B) \text{ for } 0 \leq i \leq p \text{ and } \mu_{n-k+i}(A) = \mu_i(B) \text{ for } k - q < i \leq k.$$

When we must indicate the values  $p$  and  $q$ , we say that the interlacing is  $(p, q)$ -exact.

**Problem 4** *Find conditions for  $(p, q)$ -exact interlacing.*

Among all combinations of  $p$  and  $q$ , the case of  $p = 1, q = 0$  is of primary importance. We give a solution to Problem 4 in this case, when  $A$  is nonnegative and  $B$  is a “quotient” matrix of  $A$ . Again, we introduce some notation and definitions.

Given an  $m \times n$  matrix  $A = \{a_{ij}\}$  and sets  $I \subset [m], J \subset [n]$ , write  $A[I, J]$  for the submatrix of all  $a_{ij}$  with  $i \in I$  and  $j \in J$ . A matrix  $A$  is called *regular* if its row sums are equal and so are its column sums.

Let  $A = \{a_{ij}\}$  be an  $m \times n$  matrix and let  $\mathcal{P} = \{P_1, \dots, P_k\}, \mathcal{Q} = \{Q_1, \dots, Q_l\}$  be partitions of  $[m]$  and  $[n]$ . Set  $\mathcal{P} \times \mathcal{Q} = \{P_i \times Q_j : i \in [k], j \in [l]\}$  and note that  $\mathcal{P} \times \mathcal{Q}$  is a partition of  $[m] \times [n]$ . Call the partition  $\mathcal{P} \times \mathcal{Q}$  *equitable for*  $A$  if  $A[P_p, Q_q]$  is regular for all  $p \in [k], q \in [l]$ . Write  $A|\mathcal{P} \times \mathcal{Q}$  for the  $k \times l$  matrix  $\{b_{pq}\}$  defined by

$$b_{pq} = \frac{1}{\sqrt{|P_p| |Q_q|}} \sum_{i \in P_p, j \in Q_q} a_{ij}, \quad p \in [k], q \in [l].$$

Sometimes  $A|\mathcal{P} \times \mathcal{Q}$  is called a *quotient matrix* of  $A$ .

Haemers [5] proved the following result.

**Theorem 5** *For any  $n \times n$  Hermitian matrix  $A$  and any partition  $\mathcal{P}$  of  $[n]$ , the eigenvalues of  $A$  and  $A|\mathcal{P} \times \mathcal{P}$  are interlaced; moreover, if the interlacing is tight then  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ .*

In particular, for any  $n \times n$  Hermitian matrix  $A$  and any partition  $\mathcal{P}$  of  $[n]$ ,  $\mu_1(A) \geq \mu_1(A|\mathcal{P} \times \mathcal{P})$ . We use the Perron-Frobenius theorem to prove sufficient conditions for equality in this inequality.

**Theorem 6** *If  $A$  is an irreducible, nonnegative symmetric matrix and  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ , then  $\mu_1(A) = \mu_1(A|\mathcal{P} \times \mathcal{P})$ .*

We deduce a similar result about the largest singular value of a matrix. Write  $A^*$  for the Hermitian transpose of  $A$ .

**Theorem 7** *Let  $A$  be a complex  $m \times n$  matrix,  $\mathcal{P}$  a partition of  $[m]$ , and  $\mathcal{Q}$  a partition of  $[n]$ . Then  $\sigma_1(A) \geq \sigma_1(A|\mathcal{P} \times \mathcal{Q})$ .*

*If  $A$  is nonnegative,  $AA^*$  and  $A^*A$  are irreducible, and  $\mathcal{P} \times \mathcal{Q}$  is equitable for  $A$ , then  $\sigma_1(A) = \sigma_1(A|\mathcal{P} \times \mathcal{Q})$ .*

Note that the first part of this result is implicit in [5]. Observe also that the conditions for equality in Theorem 6 and Theorem 7 are sufficient but not necessary. Thus, we have another question.

**Question 8** *For which nonnegative  $m \times n$  matrices  $A$  and partitions  $\mathcal{P}$  of  $[m]$ , and  $\mathcal{Q}$  of  $[n]$ , does the condition that  $\mathcal{P} \times \mathcal{Q}$  is equitable for  $A$  imply that  $\sigma_1(A) = \sigma_1(A|\mathcal{P} \times \mathcal{Q})$ ?*

We can answer Question 8 in a particular case, generalizing a classical result on graph spectra. Write  $G_1 + G_2$  for the join of the graphs  $G_1, G_2$  and recall a theorem of Finck and Grohmann [4] (see also [3], Theorem 2.8):

*Let the graph  $G$  be the join of an  $r_1$ -regular graph  $G_1$  of order  $n_1$  and an  $r_2$ -regular graph  $G_2$  of order  $n_2$ . Then  $\mu_1(G)$  is the positive root of the equation*

$$(x - r_1)(x - r_2) - n_1 n_2 = 0. \quad (5)$$

Setting  $\mathcal{P} = \{V(G_1), V(G_2)\}$ , a routine calculation shows that (5) is the characteristic equation of  $A(G)|\mathcal{P} \times \mathcal{P}$ ; therefore, the conclusion of the Finck-Grohmann theorem reads as

$$\mu_1(G) = \mu_1(A(G)|\mathcal{P} \times \mathcal{P}).$$

Clearly if  $G_i = G(n_i)$ ,  $2 \leq i \leq k$ , and  $G = G_1 + \dots + G_k$ , then letting  $\mathcal{P} = \{V(G_1), \dots, V(G_k)\}$ , by Theorem 5,

$$\mu_1(G) \geq \mu_1(A(G)|\mathcal{P} \times \mathcal{P}).$$

It is natural to ask when  $\mu_1(G) = \mu_1(A(G)|\mathcal{P} \times \mathcal{P})$ . We deduce the answer of this question from a more general matrix result.

**Theorem 9** *Let  $A$  be a symmetric, irreducible, nonnegative matrix of size  $n \times n$  and  $\mathcal{P} = \{P_1, \dots, P_k\}$  be a partition of  $[n]$  such that  $A[P_i, P_j]$  is regular for all  $1 \leq i < j \leq k$ . Then*

$$\mu_1(A) = \mu_1(A|\mathcal{P} \times \mathcal{P}) \quad (6)$$

*if and only if  $A[P_i, P_i]$  is regular for all  $1 \leq i \leq k$ , i.e.,  $\mathcal{P} \times \mathcal{P}$  is regular in  $A$ .*

For graphs this theorem implies the following corollary.

**Corollary 10** *Let  $G = G(n)$  be a connected graph and  $\mathcal{P}$  be a semiequitable for  $G$  partition of  $[n]$ . Then  $\mu_1(G) = \mu_1(A(G)|\mathcal{P} \times \mathcal{P})$  if and only if  $\mathcal{P}$  is equitable for  $G$ .*

## 2 Proofs

**Proof of Theorem 1** For short set  $A = A(G)$  and  $L = L(G)$ . Equality in (1) implies that

$$\mu_1(G) + \dots + \mu_k(G) = \sum_{i=1}^k \frac{2e(P_i)}{|P_i|} = \text{tr}(A|\mathcal{P} \times \mathcal{P});$$

hence  $\mu_i(G) = \mu_i(A|\mathcal{P} \times \mathcal{P})$  for all  $i \in [k]$ . Thus, the interlacing is  $k$ -tight and  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ : therefore,  $\mathcal{P}$  is equitable for  $G$ .

Inequality (2) follows from Theorem 5 and  $\mu_1(A|\mathcal{P} \times \mathcal{P}) \geq 2e(G)/n$  noting that

$$\begin{aligned} \mu_{n-k+2}(G) + \dots + \mu_n(G) &\leq \text{tr}(A|\mathcal{P} \times \mathcal{P}) - \mu_1(A|\mathcal{P} \times \mathcal{P}) \\ &= \sum_{i=1}^k \frac{2e(P_i)}{|P_i|} - \mu_1(A|\mathcal{P} \times \mathcal{P}). \end{aligned}$$

Hence, if equality holds in (2), then  $\mu_{n-k+i}(G) = \mu_i(A|\mathcal{P} \times \mathcal{P})$  for every  $i = 2, \dots, k$ . To prove that the interlacing is tight, we shall show that  $\mu_1(G) = \mu_1(A|\mathcal{P} \times \mathcal{P})$ . Note first  $\mu_1(A|\mathcal{P} \times \mathcal{P}) = 2e(G)/n$ . Also it is easy to see that the  $k$ -vector  $(\sqrt{|P_1|}, \dots, \sqrt{|P_k|})$  is an eigenvector to  $\mu_1(A|\mathcal{P} \times \mathcal{P})$ . This implies that the  $n$ -vector of all ones is an eigenvector to  $G$  and  $\mu_1(A|\mathcal{P} \times \mathcal{P})$  is an eigenvalue of  $G$ ; hence, the Perron-Frobenius theorem implies that  $G$  is regular and  $\mu_1(G) = 2e(G)/n = \mu_1(A|\mathcal{P} \times \mathcal{P})$ . Therefore, the interlacing is 1-tight and  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ ; so  $\mathcal{P}$  is equitable for  $G$ .

Inequality (3) follows from Theorem 5 and  $\mu_k(L|\mathcal{P} \times \mathcal{P}) = 0$  noting that

$$\begin{aligned} \lambda_2(G) + \dots + \lambda_k(G) &\leq \sum_{i=1}^{k-1} \mu_i(L|\mathcal{P} \times \mathcal{P}) = \text{tr}(L|\mathcal{P} \times \mathcal{P}) \\ &= \sum_{1 \leq i < j \leq k} e(P_i, P_j) \left( \frac{1}{|P_i|} + \frac{1}{|P_j|} \right). \end{aligned}$$

Consequently, by  $\lambda_1(G) = 0$ , equality in (3) implies that the interlacing is 1-tight. Hence,  $\mathcal{P} \times \mathcal{P}$  is equitable for  $L$ , and so, for all  $1 \leq i < j \leq k$ , the graphs  $G[P_i, P_j]$  are semiregular.

Finally, inequality (4) follows from Theorem 5 noting that

$$\lambda_{n-k+1}(G) + \dots + \lambda_n(G) \geq \sum_{i=1}^k \mu_i(L|\mathcal{P} \times \mathcal{P}) = \sum_{1 \leq i < j \leq k} e(P_i, P_j) \left( \frac{1}{|P_i|} + \frac{1}{|P_j|} \right).$$

Clearly, equality in (4) implies that the interlacing is  $(k-1)$ -tight. Hence,  $\mathcal{P} \times \mathcal{P}$  is equitable for  $L$ ; thus, for all  $1 \leq i < j \leq k$ , the graphs  $G[P_i, P_j]$  are semiregular, as claimed.  $\square$

**Proof of Theorem 3** For short write  $A$  for  $A(G)$ . Since  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ , for every unit eigenvector  $\mathbf{y} = (y_1, \dots, y_k)$  to an eigenvalue  $\mu$  of  $A|\mathcal{P} \times \mathcal{P}$ , the vector  $\mathbf{x} = (x_1, \dots, x_n)$  defined by

$$x_i = \frac{1}{\sqrt{|P_s|}} y_s \quad \text{for } i \in P_s$$

is a unit eigenvector of  $A$  to the eigenvalue  $\mu$ . This implies that the spectrum of  $A$  contains all eigenvalues of  $A|\mathcal{P} \times \mathcal{P}$  with the same or greater multiplicity.

On the other hand, the structure of  $G$  implies that the vertices in the same partition set  $P_i$  have the same neighbors. Thus, every eigenvalue  $\mu$  of  $A$  has an eigenvector which is constant within each  $P_i$ . This implies that every eigenvalue of  $A$  is also an eigenvalue of  $A|\mathcal{P} \times \mathcal{P}$ . Therefore,  $A$  and  $A|\mathcal{P} \times \mathcal{P}$  have the same set of eigenvalues and each eigenvalue occurs at least as many times in the spectrum of  $A$  as in the spectrum of  $A|\mathcal{P} \times \mathcal{P}$ . Since  $\text{tr}(A^2) = \text{tr}(A|\mathcal{P} \times \mathcal{P})^2$ , we see that

$$\sum_{i=1}^n \mu_i^2(A) = \sum_{i=1}^n \mu_i^2(A|\mathcal{P} \times \mathcal{P}),$$

and so  $A|\mathcal{P} \times \mathcal{P}$  and  $A$  have exactly the same nonzero eigenvalues with the same multiplicities. Hence, inequalities (1)-(4) follow immediately, completing the proof.  $\square$

**Proof of Theorem 6** Let  $\mathcal{P} = \{P_1, \dots, P_k\}$  and suppose that  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ . Since  $A$  is irreducible,  $A|\mathcal{P} \times \mathcal{P}$  is also irreducible; let  $\mathbf{y} = (y_1, \dots, y_k)$  be a positive unit eigenvector to  $\mu_1(A|\mathcal{P} \times \mathcal{P})$ . Then the vector  $\mathbf{x} = (x_1, \dots, x_n)$  defined by

$$x_i = \frac{1}{\sqrt{|P_s|}} y_s \quad \text{for } i \in P_s$$

is a positive unit vector such that  $A\mathbf{x} = \mu_1(A|\mathcal{P} \times \mathcal{P})\mathbf{x}$ , implying that  $\mu_1(A|\mathcal{P} \times \mathcal{P})$  is an eigenvalue of  $A$  with eigenvector  $\mathbf{x}$ . The Perron-Frobenius theorem implies that  $\mu_1(A|\mathcal{P} \times \mathcal{P}) = \mu_1(A)$ , completing the proof.  $\square$

**Proof of Theorem 7** For every  $i \in [l]$ , set  $Q'_i = \{x + m : x \in Q_i\}$ ; thus  $\mathcal{Q}' = \{Q'_1, \dots, Q'_l\}$  is a partition of the set  $[m+1, m+n]$  and  $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}'$  is a partition of  $[m+n]$ . Let

$$B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

It is known (see, e.g., [6], p. 418) that  $\sigma_1(A) = \mu_1(B)$ . It is easy to see that  $B$  is irreducible if and only if  $A^*A$  and  $AA^*$  are irreducible. Since

$$B|\mathcal{R} \times \mathcal{R} = \begin{pmatrix} 0 & (A|\mathcal{P} \times \mathcal{Q})^* \\ A|\mathcal{P} \times \mathcal{Q} & 0 \end{pmatrix},$$

if  $\sigma_1(A) = \sigma_1(A|\mathcal{P} \times \mathcal{Q})$ , we see that

$$\mu_1(B) = \sigma_1(A) = \sigma_1(A|\mathcal{P} \times \mathcal{Q}) = \mu_1(B|\mathbf{R} \times \mathbf{R}),$$

and Theorem 6 implies that  $\mathcal{R} \times \mathcal{R}$  is equitable for  $B$ ; hence,  $\mathcal{P} \times \mathcal{Q}$  is equitable for  $A$ , completing the proof.  $\square$

**Proof of Theorem 9** If  $A[P_i, P_i]$  is regular for each  $i \in [k]$ , the partition  $\mathcal{P} \times \mathcal{P}$  is equitable for  $A$ , and Theorem 6 implies (6). Suppose now  $\mu_1(A) = \mu_1(A|\mathcal{P} \times \mathcal{P})$ . We have to prove that  $A[P_i, P_i]$  is regular for every  $i \in [k]$ . Since  $A|\mathcal{P} \times \mathcal{P}$  is irreducible, there is a positive unit eigenvector  $\mathbf{y} = (y_1, \dots, y_k)$  to  $\mu_1(A|\mathcal{P} \times \mathcal{P})$ . Define the unit vector  $\mathbf{x} = (x_1, \dots, x_n)$  by

$$x_i = \frac{1}{\sqrt{|P_s|}} y_s \quad \text{for } i \in P_s.$$

We have  $\langle A\mathbf{x}, \mathbf{x} \rangle = \mu_1(A|\mathcal{P} \times \mathcal{P}) = \mu_1(A)$ , and so  $\mathbf{x}$  is an eigenvector of  $A$  to  $\mu_1(A)$ . For any  $r \in [k]$  and  $s, t \in P_r$ , we have

$$\begin{aligned}\mu_1(A)x_s &= \sum_{i=1}^n a_{si}x_i = \sum_{i=1}^k \frac{1}{\sqrt{|P_i|}}y_i \sum_{j \in P_i} a_{sj} \\ \mu_1(A)x_t &= \sum_{i=1}^n a_{ti}x_i = \sum_{i=1}^k \frac{1}{\sqrt{|P_i|}}y_i \sum_{j \in P_i} a_{tj}\end{aligned}$$

Since  $x_s = x_t$  and

$$\sum_{j \in P_i} a_{tj} = \sum_{j \in P_i} a_{sj}$$

for  $i \in [k] \setminus \{r\}$ , we see that

$$\sum_{j \in P_r} a_{sj} = \sum_{j \in P_r} a_{tj},$$

that is to say, the row sums of  $A[P_r, P_r]$  are equal. Since  $A[P_r, P_r]$  is symmetric, this implies that  $A[P_r, P_r]$  is regular, completing the proof.  $\square$

### Concluding remarks

In this note we confined our investigation of exact interlacing to the largest eigenvalue only. It would be good to continue this work for the smallest and the second largest eigenvalues, i.e., for  $(0, 1)$ -exact and  $(2, 0)$ -exact interlacing. Unfortunately, these important problems seem rather difficult to tackle.

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